Error estimates for the full discretization of a nonlocal parabolic model for type-I superconductors

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Abstract

A vectorial nonlocal linear parabolic problem in terms of the magnetic field for superconductors of type-I is considered. This problem is obtained from the quasi-static Maxwell equations, the two-fluid model of London and London, and the nonlocal representation of the superconductive current by Eringen (space convolution). In this contribution, a linear fully discrete approximation scheme is proposed to solve this problem. The convergence of the scheme is proved and the corresponding error estimates are derived under appropriate assumptions. It is also shown how to improve the error estimates under higher regularity.

Keywords: Maxwell equations, nonlocal superconductors, singular convolution kernel, full discretization, error estimates

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1. Introduction

In this contribution, a superconductive material of type-I occupies a bounded polyhedral Lipschitz continuous domain $\Omega \subset \mathbb{R}^3$, with boundary $\partial\Omega$. The symbol $\boldsymbol{\nu}$ denotes the outward unit normal vector on $\partial\Omega$. In [1], the authors proposed the following model in terms of the magnetic field $\boldsymbol{H}$ for type-I superconductors:

$$\partial_t \boldsymbol{H} + \nabla \times \nabla \times \boldsymbol{H} + \nabla \times (\mathcal{K}_0 \star \boldsymbol{H}) = \boldsymbol{f}. \tag{1}$$

This equation is obtained from the quasi-static Maxwell equations, the two-fluid model of London and London, and the nonlocal representation of the superconductive current by Eringen [2, 3, 4]. In the two-fluid model, the current density $\boldsymbol{J}$ is supposed to be the sum of a normal ($\boldsymbol{J}_n$) and a superconducting part ($\boldsymbol{J}_s$). The normal density current $\boldsymbol{J}_n$ is satisfying Ohm's law. For the superconductive part of the current $\boldsymbol{J}_s$, the nonlocal representation of the superconductive current by Eringen is considered [4]. This representation identifies the state of the superconductor at time t with the field $\boldsymbol{H}(\cdot, t)$ and is given by the linear functional

$$\boldsymbol{J}_s(\boldsymbol{x}, t) = \int_\Omega \sigma_0\left(|\boldsymbol{x} - \boldsymbol{x}'|\right)(\boldsymbol{x} - \boldsymbol{x}') \times \boldsymbol{H}(\boldsymbol{x}', t)\,\mathrm{d}\boldsymbol{x}' =: -(\mathcal{K}_0 \star \boldsymbol{H})(\boldsymbol{x}, t), \quad (\boldsymbol{x}, t) \in \Omega \times (0, T),$$

where the singular kernel $\sigma_0 : (0, \infty) \to \mathbb{R}$ is defined by

$$\sigma_0(s) = \begin{cases} \frac{\widetilde{C}}{2s^2} \exp\left(-\frac{s}{r_0}\right) & s < r_0; \\ 0 & s \geqslant r_0, \end{cases}$$

with $\widetilde{C} := \frac{3}{4\pi\xi_0\Lambda} > 0$. The length ξ_0 is called the coherence length of the material and $\Lambda = \frac{m_e}{n_s e^2}$, with n_s the number of superelectrons per unit volume, m_e and $-e$ the mass and the electric charge of an electron respectively. The points

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which contribute to the integral are separated by distances of order r_0 or less, where $r_0 = \frac{\xi_0 l}{\xi_0 + l}$, with l the mean free path of the electrons in the material.

In [1], only the well-posedness of problem (1) is shown under low regularity assumptions. Also error estimates are derived for different time-discrete schemes. The aim of this paper is to design a fully discrete finite element scheme to approximate the solution of the following vectorial nonlocal linear parabolic problem

$$\begin{cases} \partial_t \boldsymbol{H} + \nabla\times\nabla\times\boldsymbol{H} + \nabla\times(\mathcal{K}_0 \star \boldsymbol{H}) &= \boldsymbol{f} \quad \text{in } Q_T := \Omega\times(0,T); \\ \boldsymbol{H}\times\boldsymbol{\nu} &= \boldsymbol{0} \quad \text{on } \partial\Omega\times(0,T); \\ \boldsymbol{H}(\boldsymbol{x},0) = \boldsymbol{H}_0, \quad \nabla\cdot\boldsymbol{H}_0 &= 0 \quad \text{in } \Omega. \end{cases} \tag{2}$$

The outline of this paper is as follows. In Section 2, the mathematical tools are summarized. A time-discrete scheme is described in Section 3. In Section 4, a fully discrete finite element scheme is proposed to approximate the solution to problem (2). Moreover, the error estimates for the full discretization are derived. Under higher regularity assumptions, better error estimates are derived in Section 5.

2. Functional setting

In this section, some standard notations are introduced. The Euclidian norm of a vector $\boldsymbol{v}$ in $\mathbb{R}^3$ is expressed by $|\boldsymbol{v}|$. The Lebesgue spaces of vector-valued functions with componentwise p-th power integrable functions are denoted by $\mathbf{L}^p(\Omega)$ with the usual norm $\|\cdot\|_p$. For instance, in the special case $p = 2$, the $\mathbf{L}^2(\Omega)$ scalar product is denoted by $(\boldsymbol{u},\boldsymbol{v}) = \int_\Omega \boldsymbol{u}\cdot\boldsymbol{v}\,\mathrm{d}\boldsymbol{x}$ and the corresponding norm is $\|\boldsymbol{v}\| = \sqrt{(\boldsymbol{v},\boldsymbol{v})}$. The following spaces are used in our analysis: $\mathbf{H}^1(\Omega)$, $\mathbf{H}^2(\Omega)$, $\mathbf{H}(\mathbf{curl},\Omega)$ and the fractional Sobolev spaces $\mathbf{H}^s(\Omega)$ and $\mathbf{H}^s(\mathbf{curl},\Omega)$ – see [5]. The Hilbert space $\mathbf{H}^1(\Omega)$ is endowed with the norm $\|\boldsymbol{\varphi}\|^2_{\mathbf{H}^1(\Omega)} = \|\boldsymbol{\varphi}\|^2 + \|\nabla\boldsymbol{\varphi}\|^2$. The space $\mathbf{H}(\mathbf{curl},\Omega)$ is a Banach space with respect to the graph norm $\|\boldsymbol{\varphi}\|^2_{\mathbf{H}(\mathbf{curl},\Omega)} = \|\boldsymbol{\varphi}\|^2 + \|\nabla\times\boldsymbol{\varphi}\|^2$. The spaces $\mathbf{H}^1_0(\Omega)$ and $\mathbf{H}_0(\mathbf{curl},\Omega)$ inherit the norm $\|\boldsymbol{\varphi}\|_{\mathbf{H}^1(\Omega)}$ and $\|\boldsymbol{\varphi}\|_{\mathbf{H}(\mathbf{curl},\Omega)}$, respectively. The space of Lipschitz continuous functions $\boldsymbol{f} : [0,T] \to \mathbf{L}^2(\Omega)$ is denoted by $\mathrm{Lip}\left([0,T],\mathbf{L}^2(\Omega)\right)$.

The values C, ε and C_ε are generic and positive constants independent of the discretization parameters τ and h. The value ε is small and $C_\varepsilon = C\left(\varepsilon^{-1}\right)$. To reduce the number of arbitrary constants, the notation $a \lesssim b$ is used if there exists a constant C such that $a \leqslant Cb$.

2.1. Useful estimates

In this subsection, some useful estimates that are crucial in the calculations are derived. Using spherical coordinates one can deduce that $\sigma_0(|\boldsymbol{x}|)\boldsymbol{x}$ belongs to $\mathbf{L}^p(\Omega)$ for $1 \leqslant p < 3$. Hence, the following estimates on $\boldsymbol{J}_s$ can be obtained

$$|\boldsymbol{J}_s(\boldsymbol{x},t)| = |(\mathcal{K}_0 \star \boldsymbol{H})(\boldsymbol{x},t)| \leqslant C(q)\,\|\boldsymbol{H}(t)\|_q\,, \qquad q > \frac{3}{2}, \qquad \forall \boldsymbol{x}\in\Omega. \tag{3}$$

Therefore, using Young's inequality, it is for instance true that

$$(\mathcal{K}_0 \star \boldsymbol{h}_1, \nabla\times\boldsymbol{h}_2) \overset{(3)}{\leqslant} C_\varepsilon\,\|\boldsymbol{h}_1\|^2 + \varepsilon\,\|\nabla\times\boldsymbol{h}_2\|^2\,, \qquad \forall \boldsymbol{h}_1 \in \mathbf{L}^2(\Omega), \boldsymbol{h}_2 \in \mathbf{H}_0(\mathbf{curl},\Omega). \tag{4}$$

3. Time discretization

Applying the semidiscretization in time, the existence of a solution to (2) is proved in [1]. This discretization is based on backward Euler (Rothe's) method [6]. The interval $[0,T]$ is divided into n equidistant subintervals $[t_{i-1}, t_i]$ with time step $\tau = \frac{T}{n} < 1$, thus $t_i = i\tau, i = 0,\dots,n$. The following standard notations for the discretized fields are introduced: $\boldsymbol{h}_i \approx \boldsymbol{H}(t_i)$ and $\delta\boldsymbol{h}_i = \frac{\boldsymbol{h}_i - \boldsymbol{h}_{i-1}}{\tau}$. The variational formulation of (2) reads as

$$(\partial_t \boldsymbol{H}, \boldsymbol{\varphi}) + (\nabla\times\boldsymbol{H}, \nabla\times\boldsymbol{\varphi}) + (\mathcal{K}_0 \star \boldsymbol{H}, \nabla\times\boldsymbol{\varphi}) = (\boldsymbol{f},\boldsymbol{\varphi})\,, \qquad \forall\boldsymbol{\varphi}\in\mathbf{H}_0(\mathbf{curl},\Omega). \tag{5}$$

The following linear recurrent scheme is proposed in [1] to approximate this problem

$$\begin{cases} (\delta\boldsymbol{h}_i, \boldsymbol{\varphi}) + (\nabla\times\boldsymbol{h}_i, \nabla\times\boldsymbol{\varphi}) + (\mathcal{K}_0 \star \boldsymbol{h}_i, \nabla\times\boldsymbol{\varphi}) &= (\boldsymbol{f}_i, \boldsymbol{\varphi})\,, \qquad \boldsymbol{\varphi}\in\mathbf{H}_0(\mathbf{curl},\Omega); \\ \boldsymbol{h}_0 &= \boldsymbol{H}_0. \end{cases} \tag{6}$$

The vector fields $\boldsymbol{h}_n$ and $\overline{\boldsymbol{h}}_n$ are defined

$$\begin{aligned}
\boldsymbol{h}_n(0) &= \boldsymbol{H}_0; \quad \boldsymbol{h}_n(t) = \boldsymbol{h}_{i-1} + (t - t_{i-1})\delta \boldsymbol{h}_i \qquad && \text{for } t \in (t_{i-1}, t_i], \quad i = 1, \dots, n;\\
\overline{\boldsymbol{h}}_n(0) &= \boldsymbol{H}_0, \quad \overline{\boldsymbol{h}}_n(t) = \boldsymbol{h}_i, && \text{for } t \in (t_{i-1}, t_i], \quad i = 1, \dots, n.
\end{aligned}$$

Similary, the vector field $\overline{\boldsymbol{f}}_n$ is defined. Using this notations, the variational formulation (6) can be rewritten as

$$(\partial_t \boldsymbol{h}_n(t), \boldsymbol{\varphi}) + \left(\nabla \times \overline{\boldsymbol{h}}_n(t), \nabla \times \boldsymbol{\varphi}\right) + \left(\mathcal{K}_0 \star \overline{\boldsymbol{h}}_n(t), \nabla \times \boldsymbol{\varphi}\right) = \left(\overline{\boldsymbol{f}}_n(t), \boldsymbol{\varphi}\right). \tag{7}$$

The convergence of the proposed approximation scheme is shown in [1] and also error estimates for the time discretization are derived. The most important results are summarized in the following theorem.

Theorem 1 (Existence and uniqueness)**.**

- *Let $\boldsymbol{H}_0 \in \mathbf{L}^2(\Omega)$ and $\boldsymbol{f} \in L^2\left((0,T), \mathbf{L}^2(\Omega)\right)$. Assume that $\nabla \cdot \boldsymbol{H}_0 = 0 = \nabla \cdot \boldsymbol{f}(t)$ for any time $t \in [0,T]$. Then there exists a unique solution $\boldsymbol{H} \in C\left([0,T], \mathbf{L}^2(\Omega)\right) \cap L^2\left((0,T), \mathbf{H}^{\frac{1}{2}}(\Omega)\right)$ with $\partial_t \boldsymbol{H} \in L_2\left((0,T), \mathbf{H}_0^{-1}(\mathbf{curl}, \Omega)\right)$, which solves (5). If $\boldsymbol{H}_0 \in \mathbf{H}_0(\mathbf{curl}, \Omega)$, then $\partial_t \boldsymbol{H} \in L_2\left((0,T), \mathbf{L}^2(\Omega)\right)$.*

- *Suppose that $\boldsymbol{f} \in \mathrm{Lip}\left([0,T], \mathbf{L}^2(\Omega)\right)$.*

  (i) If $\boldsymbol{H}_0 \in \mathbf{H}_0(\mathbf{curl}, \Omega)$ then

$$\max_{t \in [0,T]} \|\boldsymbol{h}_n(t) - \boldsymbol{H}(t)\|^2 + \int_0^T \|\nabla \times [\boldsymbol{h}_n - \boldsymbol{H}]\|^2 \leqslant C\tau.$$

  (ii) If $\nabla \times (\mathcal{K}_0 \star \boldsymbol{H}_0) \in \mathbf{L}^2(\Omega)$, $\boldsymbol{H}_0 \in \mathbf{H}_0(\mathbf{curl}, \Omega)$ and $\nabla \times \nabla \times \boldsymbol{H}_0 \in \mathbf{L}^2(\Omega)$ then

$$\max_{t \in [0,T]} \|\boldsymbol{h}_n(t) - \boldsymbol{H}(t)\|^2 + \int_0^T \|\nabla \times [\boldsymbol{h}_n - \boldsymbol{H}]\|^2 \leqslant C\tau^2.$$

  Please note that the positive constant C in this estimates is of the form Ce^{CT}.

4. A fully discrete finite element scheme

In this section, a linear numerical scheme discretized in time and space for finding an approximation of the solution to problem (2) is suggested. The purpose of the finite element method is to approximate the solution of a problem in a finite dimensional space. The first step is to generate a finite element mesh that covers the domain Ω. The domain Ω can be subdivided into a finite set of distinct thetrahedra $\mathcal{T} = \{K\}$ such that $\cup_{K \in \mathcal{T}} \overline{K} = \overline{\Omega}$, see [7]. In our analysis, it is assumed that there is a regular family of meshes or triangulations $\{\mathcal{T}^h : h > 0\}$, where h denotes the mesh parameter. The purpose of this paper is to analyze the error as h decreases. The second step is the consideration of a finite element subspace $\boldsymbol{V}^h$ of $\mathbf{H}(\mathbf{curl}, \Omega)$. In order to take the boundary condition $\boldsymbol{H} \times \boldsymbol{\nu} = \mathbf{0}$ into account, the finite dimensional subspace $\boldsymbol{V}_0^h = \{\boldsymbol{v}^h \in \boldsymbol{V}^h : \boldsymbol{v}^h \times \boldsymbol{\nu} = \mathbf{0} \text{ on } \partial\Omega\}$ of $\mathbf{H}_0(\mathbf{curl}, \Omega)$ is considered. Let $\boldsymbol{P}_h : \mathbf{L}^2(\Omega) \to \boldsymbol{V}_0^h$ the orthogonal projection operator such that if $\boldsymbol{u} \in \mathbf{L}^2(\Omega)$ then $\boldsymbol{P}_h \boldsymbol{u} \in \boldsymbol{V}_0^h$ satisfies

$$(\boldsymbol{u}, \boldsymbol{v}_h) = (\boldsymbol{P}_h \boldsymbol{u}, \boldsymbol{v}_h), \qquad \forall \boldsymbol{v}_h \in \boldsymbol{V}_0^h. \tag{8}$$

Analogously, let $\widetilde{\boldsymbol{P}}_h : \mathbf{H}_0(\mathbf{curl}, \Omega) \to \boldsymbol{V}_0^h$ the orthogonal projection operator such that if $\boldsymbol{u} \in \mathbf{H}_0(\mathbf{curl}, \Omega)$ then $\widetilde{\boldsymbol{P}}_h \boldsymbol{u} \in \boldsymbol{V}_0^h$ satisfies

$$(\boldsymbol{u}, \boldsymbol{v}_h) + (\nabla \times \boldsymbol{u}, \nabla \times \boldsymbol{v}_h) = \left(\widetilde{\boldsymbol{P}}_h \boldsymbol{u}, \boldsymbol{v}_h\right) + \left(\nabla \times \widetilde{\boldsymbol{P}}_h \boldsymbol{u}, \nabla \times \boldsymbol{v}_h\right), \qquad \forall \boldsymbol{v}_h \in \boldsymbol{V}_0^h. \tag{9}$$

Choosing $\boldsymbol{v}_h = \boldsymbol{P}_h \boldsymbol{u}$ in (8) and $\boldsymbol{v}_h = \widetilde{\boldsymbol{P}}_h \boldsymbol{u}$ in (9), it is easy to proof that $\boldsymbol{P}_h$ and $\widetilde{\boldsymbol{P}_h}$ are linear bounded operators.

At this point, a fully discrete scheme can be defined. After time and space discretization, the following approximation of our problem can be obtained: find $\boldsymbol{h}_i^h \in \boldsymbol{V}_0^h$ such that

$$\begin{cases} \left(\delta \boldsymbol{h}_i^h, \boldsymbol{\varphi}^h\right) + \left(\nabla \times \boldsymbol{h}_i^h, \nabla \times \boldsymbol{\varphi}^h\right) + \left(\mathcal{K}_0 \star \boldsymbol{h}_i^h, \nabla \times \boldsymbol{\varphi}^h\right) & = \left(\boldsymbol{P}_h \boldsymbol{f}_i, \boldsymbol{\varphi}^h\right) = \left(\boldsymbol{f}_i, \boldsymbol{\varphi}^h\right); \\ \boldsymbol{h}_0^h & = \widetilde{\boldsymbol{P}}_h \boldsymbol{H}_0, \end{cases} \tag{10}$$

is satisfied for all $\boldsymbol{\varphi}^h \in \boldsymbol{V}_0^h$. This problem is equivalent with solving $a^h(\boldsymbol{h}_i^h, \boldsymbol{\varphi}^h) = f^h(\boldsymbol{\varphi}^h)$ for all $\boldsymbol{\varphi}^h \in \boldsymbol{V}_0^h$, where $a^h : \boldsymbol{V}_0^h \times \boldsymbol{V}_0^h \to \mathbb{R}$ and $f^h : \boldsymbol{V}_0^h \to \mathbb{R}$ are defined by

$$a^h(\boldsymbol{h}_i^h, \boldsymbol{\varphi}^h) = \left(\frac{\boldsymbol{h}_i^h}{\tau}, \boldsymbol{\varphi}^h\right) + \left(\nabla \times \boldsymbol{h}_i^h, \nabla \times \boldsymbol{\varphi}^h\right) + \left(\mathcal{K}_0 \star \boldsymbol{h}_i^h, \nabla \times \boldsymbol{\varphi}^h\right) \quad \text{and} \quad f^h(\boldsymbol{\varphi}^h) = \left(\boldsymbol{f}_i, \boldsymbol{\varphi}^h\right) + \left(\frac{\boldsymbol{h}_{i-1}^h}{\tau}, \boldsymbol{\varphi}^h\right).$$

Remark that $\boldsymbol{h}_i^h$ denotes the finite element solution at time $t = t_i$.

Theorem 2. *Suppose that $\boldsymbol{H}_0 \in \mathbf{H}_0(\mathbf{curl}\,, \Omega)$. Then the variational problem (10) admits a unique solution $\boldsymbol{h}_i^h \in \boldsymbol{V}_0^h$ for any $i = 1, \ldots, n$ if $\tau < \tau_0$.*

Proof. This is an easy application of the Lax-Milgram lemma for any $i = 1, \ldots, n$. It holds that

$$a^h(\boldsymbol{v}^h, \boldsymbol{v}^h) \geqslant \frac{1}{\tau} \left\|\boldsymbol{v}^h\right\|^2 + \left\|\nabla \times \boldsymbol{v}^h\right\|^2 - \left|\left(\mathcal{K}_0 \star \boldsymbol{v}^h, \nabla \times \boldsymbol{v}^h\right)\right| \overset{(4)}{\geqslant} \left(\frac{1}{\tau} - C_\varepsilon\right) \left\|\boldsymbol{v}^h\right\|^2 + (1 - \varepsilon) \left\|\nabla \times \boldsymbol{v}^h\right\|^2.$$

Fixing $\varepsilon < 1$, proofs that the bilinear form $a^h(\cdot, \cdot)$ is elliptic in the Hilbert space $\boldsymbol{V}_0^h$ for $\tau < \tau_0$. Moreover, a^h is continuous in $\boldsymbol{V}_0^h$. If $\boldsymbol{H}_0 \in \mathbf{H}_0(\mathbf{curl}\,, \Omega)$, then the functional $f^h(\cdot)$ is linear and bounded in $\boldsymbol{V}_0^h$. □

A stability analysis is needed to derive the error estimates for the full discretization.

Lemma 1 (Stability analysis)**.** *Suppose that $\boldsymbol{f} \in L^2\left((0, T), \mathbf{L}^2(\Omega)\right)$.*

(i) Let $\boldsymbol{H}_0 \in \mathbf{H}_0(\mathbf{curl}\,, \Omega)$. Then, there exists a positive constant C such that for all $\tau < \tau_0$

$$\max_{1 \leqslant i \leqslant n} \left\|\boldsymbol{h}_i^h\right\|^2 + \sum_{i=1}^n \left\|\boldsymbol{h}_i^h - \boldsymbol{h}_{i-1}^h\right\|^2 + \sum_{i=1}^n \left\|\nabla \times \boldsymbol{h}_i^h\right\|^2 \tau \leqslant C.$$

(ii) If $\boldsymbol{H}_0 \in \mathbf{H}_0(\mathbf{curl}\,, \Omega)$ then for all $\tau < \tau_0$

$$\max_{1 \leqslant i \leqslant n} \left\|\nabla \times \boldsymbol{h}_i^h\right\|^2 + \sum_{i=1}^n \left\|\nabla \times \boldsymbol{h}_i^h - \nabla \times \boldsymbol{h}_{i-1}^h\right\|^2 + \sum_{i=1}^n \left\|\delta \boldsymbol{h}_i^h\right\|^2 \tau \leqslant C.$$

(iii) If $\boldsymbol{f}(0) \in \mathbf{L}^2(\Omega)$, $\partial_t \boldsymbol{f} \in L^2\left((0, T), \mathbf{L}^2(\Omega)\right)$, $\nabla \times (\mathcal{K}_0 \star \boldsymbol{H}_0) \in \mathbf{L}^2(\Omega)$, $\boldsymbol{H}_0 \in \mathbf{H}_0(\mathbf{curl}\,, \Omega)$ and $\nabla \times \nabla \times \boldsymbol{H}_0 \in \mathbf{L}^2(\Omega)$ then for all $\tau < \tau_0$

$$\max_{1 \leqslant i \leqslant n} \left\|\delta \boldsymbol{h}_i^h\right\|^2 + \sum_{i=1}^n \left\|\delta \boldsymbol{h}_i^h - \delta \boldsymbol{h}_{i-1}^h\right\|^2 + \sum_{i=1}^n \left\|\nabla \times \delta \boldsymbol{h}_i^h\right\|^2 \tau \leqslant C.$$

Proof. (i) First, we set $\boldsymbol{\varphi}^h = \boldsymbol{h}_i^h$ in (10). Then, we multiply the result by τ and sum up for $i = 1, \ldots, j$ to arrive at

$$\sum_{i=1}^j \left(\delta \boldsymbol{h}_i^h, \boldsymbol{h}_i^h\right) \tau + \sum_{i=1}^j \left\|\nabla \times \boldsymbol{h}_i^h\right\|^2 \tau + \sum_{i=1}^j \left(\mathcal{K}_0 \star \boldsymbol{h}_i^h, \nabla \times \boldsymbol{h}_i^h\right) \tau = \sum_{i=1}^j \left(\boldsymbol{f}_i, \boldsymbol{h}_i^h\right) \tau.$$

For the first term on the left-hand side (LHS), we use Abel's summation rule

$$2 \sum_{i=1}^j \left(\delta \boldsymbol{h}_i^h, \boldsymbol{h}_i^h\right) \tau = \left\|\boldsymbol{h}_j^h\right\|^2 - \left\|\widetilde{\boldsymbol{P}}_h \boldsymbol{H}_0\right\|^2 + \sum_{i=1}^j \left\|\boldsymbol{h}_i^h - \boldsymbol{h}_{i-1}^h\right\|^2,$$

For the third term on the LHS, we have using (4) that

$$\left|\sum_{i=1}^{j}\left(\mathcal{K}_0 \star \boldsymbol{h}_i^h, \nabla\times\boldsymbol{h}_i^h\right)\tau\right| \leqslant \varepsilon\sum_{i=1}^{j}\left\|\nabla\times\boldsymbol{h}_i^h\right\|^2\tau + C_\varepsilon\sum_{i=1}^{j}\left\|\boldsymbol{h}_i^h\right\|^2\tau.$$

For the RHS, we apply the Cauchy and Young inequalities. Fixing ε sufficiently small and applying the Grönwall argument gives the proof.

(*ii*) Now, we put $\boldsymbol{\varphi}^h = \delta\boldsymbol{h}_i^h$ in (10). Again, we multiply by τ and sum up for $i = 1, \dots, j$

$$\sum_{i=1}^{j}\left\|\delta\boldsymbol{h}_i^h\right\|^2\tau + \sum_{i=1}^{j}\left(\nabla\times\boldsymbol{h}_i^h, \nabla\times\boldsymbol{h}_i^h - \nabla\times\boldsymbol{h}_{i-1}^h\right) + \sum_{i=1}^{j}\left(\mathcal{K}_0\star\boldsymbol{h}_i^h, \nabla\times\delta\boldsymbol{h}_i^h\right)\tau = \sum_{i=1}^{j}\left(\boldsymbol{f}_i, \delta\boldsymbol{h}_i^h\right)\tau.$$

Abel's summation rule gives

$$2\sum_{i=1}^{j}\left(\nabla\times\boldsymbol{h}_i^h, \nabla\times\boldsymbol{h}_i^h - \nabla\times\boldsymbol{h}_{i-1}^h\right) = \left\|\nabla\times\boldsymbol{h}_j^h\right\|^2 - \left\|\nabla\times\widetilde{\boldsymbol{P}}_h\boldsymbol{H}_0\right\|^2 + \sum_{i=1}^{j}\left\|\nabla\times\boldsymbol{h}_i^h - \nabla\times\boldsymbol{h}_{i-1}^h\right\|^2$$

and

$$\sum_{i=1}^{j}\left(\mathcal{K}_0\star\boldsymbol{h}_i^h, \nabla\times\delta\boldsymbol{h}_i^h\right)\tau = \left(\mathcal{K}_0\star\boldsymbol{h}_j^h, \nabla\times\boldsymbol{h}_j^h\right) - \left(\mathcal{K}_0\star\widetilde{\boldsymbol{P}}_h\boldsymbol{H}_0, \nabla\times\widetilde{\boldsymbol{P}}_h\boldsymbol{H}_0\right) - \sum_{i=1}^{j}\left(\mathcal{K}_0\star\delta\boldsymbol{h}_i^h, \nabla\times\boldsymbol{h}_{i-1}^h\right)\tau.$$

Hence, using (*i*) and (4), we obtain

$$\left|\sum_{i=1}^{j}\left(\mathcal{K}_0\star\boldsymbol{h}_i^h, \nabla\times\delta\boldsymbol{h}_i^h\right)\tau\right| \leqslant C_\varepsilon + \varepsilon\left\|\nabla\times\boldsymbol{h}_j^h\right\|^2 + \varepsilon\sum_{i=1}^{j}\left\|\delta\boldsymbol{h}_i^h\right\|^2\tau.$$

Combining all the estimates and fixing a sufficiently small positive ε conclude the proof.

(*iii*) We define the following compatibility condition

$$\delta\boldsymbol{h}_0^h := \boldsymbol{P}_h\boldsymbol{f}(0) - \boldsymbol{P}_h\left(\nabla\times\nabla\times\boldsymbol{H}_0\right) - \boldsymbol{P}_h\left(\nabla\times\left(\mathcal{K}_0\star\boldsymbol{H}_0\right)\right).$$

We subtract (10) for $i = i-1$ from (10), then we set $\boldsymbol{\varphi}^h = \delta\boldsymbol{h}_i^h$ and we sum the result for $i = 1, \dots, j$ with $1 \leqslant j \leqslant n$ to get

$$\sum_{i=1}^{j}\left(\delta^2\boldsymbol{h}_i^h, \delta\boldsymbol{h}_i^h\right)\tau + \sum_{i=1}^{j}\left\|\nabla\times\delta\boldsymbol{h}_i^h\right\|^2\tau + \sum_{i=1}^{j}\left(\mathcal{K}_0\star\delta\boldsymbol{h}_i^h, \nabla\times\delta\boldsymbol{h}_i^h\right)\tau = \sum_{i=1}^{j}\left(\delta\boldsymbol{f}_i, \delta\boldsymbol{h}_i^h\right)\tau.$$

Further, we follow the same lines as in (*i*) when considering $\delta\boldsymbol{h}_i^h$ instead of $\boldsymbol{h}_i^h$. □

Now, the following piecewise linear in time vector fields $\boldsymbol{h}_n^h$ and the piecewise constant in time fields $\overline{\boldsymbol{h}}_n^h$ are defined

$$\begin{array}{llll} \boldsymbol{h}_n^h(0) & = \widetilde{\boldsymbol{P}}_h\boldsymbol{H}_0, \quad \boldsymbol{h}_n^h(t) = \boldsymbol{h}_{i-1}^h + (t - t_{i-1})\delta\boldsymbol{h}_i^h & \text{for } t \in (t_{i-1}, t_i], & i = 1, \dots, n \\ \overline{\boldsymbol{h}}_n^h(0) & = \widetilde{\boldsymbol{P}}_h\boldsymbol{H}_0, \quad \overline{\boldsymbol{h}}_n^h(t) = \boldsymbol{h}_i^h, & \text{for } t \in (t_{i-1}, t_i], & i = 1, \dots, n. \end{array}$$

The full discretized system (10) can be rewritten by Rothe's notation as follows

$$\left\{\begin{array}{rll} \left(\partial_t\boldsymbol{h}_n^h(t), \boldsymbol{\varphi}^h\right) + \left(\nabla\times\overline{\boldsymbol{h}}_n^h(t), \nabla\times\boldsymbol{\varphi}^h\right) + \left(\mathcal{K}_0\star\overline{\boldsymbol{h}}_n^h(t), \nabla\times\boldsymbol{\varphi}^h\right) & = \left(\overline{\boldsymbol{f}}_n(t), \boldsymbol{\varphi}^h\right), & \forall\boldsymbol{\varphi}^h \in \boldsymbol{V}_0^h; \\ \boldsymbol{h}_n^h(0) & = \widetilde{\boldsymbol{P}}_h\boldsymbol{H}_0. & \end{array}\right. \tag{11}$$

The next theorem summarizes the error estimate for the full discretization.

Theorem 3. *Suppose that* $\boldsymbol{f} \in \mathrm{Lip}([0,T], \mathbf{L}^2(\Omega))$.

(i) Let the weak solution $\boldsymbol{H}$ of (2) at time t and the initial condition $\boldsymbol{H}_0$ satisfy $\boldsymbol{H}(t), \boldsymbol{H}_0 \in \mathbf{H}_0(\mathbf{curl}\,, \Omega)$. Then for any $\tau < \tau_0$, there exists a constant C such that

$$\left\|\boldsymbol{H}(\eta) - \boldsymbol{h}_n^h(\eta)\right\|^2 + \int_0^\eta \left\|\nabla \times \left(\boldsymbol{H} - \overline{\boldsymbol{h}}_n^h\right)\right\|^2$$
$$\leqslant C\left(\tau + \left\|\boldsymbol{H}_0 - \widetilde{\boldsymbol{P}}_h\boldsymbol{H}_0\right\|^2 + \sqrt{\int_0^\eta \left\|\boldsymbol{H} - \widetilde{\boldsymbol{P}}_h\boldsymbol{H}\right\|^2} + \int_0^\eta \left\|\boldsymbol{H} - \widetilde{\boldsymbol{P}}_h\boldsymbol{H}\right\|^2 + \int_0^\eta \left\|\nabla \times \left(\boldsymbol{H} - \widetilde{\boldsymbol{P}}_h\boldsymbol{H}\right)\right\|^2\right),$$

is valid for any $\eta \in (0, T)$;

(ii) Let the weak solution $\boldsymbol{H}$ of (2) at time t and the initial condition $\boldsymbol{H}_0$ satisfy $\boldsymbol{H}(t), \partial_t\boldsymbol{H}(t), \boldsymbol{H}_0 \in \mathbf{H}_0(\mathbf{curl}\,, \Omega)$. Then for any $\tau < \tau_0$, there exists a constant C such that

$$\left\|\boldsymbol{H}(\eta) - \boldsymbol{h}_n^h(\eta)\right\|^2 + \int_0^\eta \left\|\nabla \times \left(\boldsymbol{H} - \overline{\boldsymbol{h}}_n^h\right)\right\|^2 \leqslant C\left(\tau + \left\|\boldsymbol{H}_0 - \widetilde{\boldsymbol{P}}_h\boldsymbol{H}_0\right\|^2 + \left\|\boldsymbol{H}(\eta) - \widetilde{\boldsymbol{P}}_h\boldsymbol{H}(\eta)\right\|^2\right.$$
$$\left. + \int_0^\eta \left\|\boldsymbol{H} - \widetilde{\boldsymbol{P}}_h\boldsymbol{H}\right\|^2 + \int_0^\eta \left\|\partial_t\left(\boldsymbol{H} - \widetilde{\boldsymbol{P}}_h\boldsymbol{H}\right)\right\|^2 + \int_0^\eta \left\|\nabla \times \left(\boldsymbol{H} - \widetilde{\boldsymbol{P}}_h\boldsymbol{H}\right)\right\|^2\right),$$

is valid for any $\eta \in (0, T)$;

(iii) If the initial condition satisfies $\nabla \times \boldsymbol{H}_0$ and $\mathcal{K}_0 \star \boldsymbol{H}_0 \in \mathbf{H}_0(\mathbf{curl}\,, \Omega)$, then the estimates in (i) and (ii) are satisfied with τ^2 instead of τ.

Please note that the positive constant C in this estimates is of the form Ce^{CT}.

Proof. (i) We subtract (11) from (5) for $\boldsymbol{\varphi} = \boldsymbol{\varphi}^h$. We set $\boldsymbol{\varphi}^h = \widetilde{\boldsymbol{P}}_h\boldsymbol{H}(t) - \boldsymbol{h}_n^h(t)$ and integrate in time over $(0, \eta)$ for $\eta \in [0, T]$ to get

$$\int_0^\eta \left(\partial_t\boldsymbol{H} - \partial_t\boldsymbol{h}_n^h, \widetilde{\boldsymbol{P}}_h\boldsymbol{H} - \boldsymbol{h}_n^h\right) + \int_0^\eta \left(\nabla \times \left(\boldsymbol{H} - \overline{\boldsymbol{h}}_n^h\right), \nabla \times \left(\widetilde{\boldsymbol{P}}_h\boldsymbol{H} - \boldsymbol{h}_n^h\right)\right)$$
$$+ \int_0^\eta \left(\mathcal{K}_0 \star \left(\boldsymbol{H} - \overline{\boldsymbol{h}}_n^h\right), \nabla \times \left(\widetilde{\boldsymbol{P}}_h\boldsymbol{H} - \boldsymbol{h}_n^h\right)\right) = \int_0^\eta \left(\boldsymbol{f} - \overline{\boldsymbol{f}}_n, \widetilde{\boldsymbol{P}}_h\boldsymbol{H} - \boldsymbol{h}_n^h\right).$$

We rearrange the terms by adding $\pm\boldsymbol{H}$ and $\pm\overline{\boldsymbol{h}}_n^h$ to obtain

$$\frac{1}{2}\left\|\boldsymbol{H}(\eta) - \boldsymbol{h}_n^h(\eta)\right\|^2 - \frac{1}{2}\left\|\boldsymbol{H}_0 - \widetilde{\boldsymbol{P}}_h\boldsymbol{H}_0\right\|^2 + \int_0^\eta \left\|\nabla \times \left(\boldsymbol{H} - \overline{\boldsymbol{h}}_n^h\right)\right\|^2$$
$$= \int_0^\eta \left(\partial_t\boldsymbol{H} - \partial_t\boldsymbol{h}_n^h, \boldsymbol{H} - \widetilde{\boldsymbol{P}}_h\boldsymbol{H}\right) + \int_0^\eta \left(\nabla \times \left(\boldsymbol{H} - \overline{\boldsymbol{h}}_n^h\right), \nabla \times \left(\boldsymbol{H} - \widetilde{\boldsymbol{P}}_h\boldsymbol{H}\right)\right) + \int_0^\eta \left(\nabla \times \left(\boldsymbol{H} - \overline{\boldsymbol{h}}_n^h\right), \nabla \times \left(\boldsymbol{h}_n^h - \overline{\boldsymbol{h}}_n^h\right)\right)$$
$$+ \int_0^\eta \left(\mathcal{K}_0 \star \left(\boldsymbol{H} - \overline{\boldsymbol{h}}_n^h\right), \nabla \times \left(\boldsymbol{H} - \widetilde{\boldsymbol{P}}_h\boldsymbol{H}\right)\right) + \int_0^\eta \left(\mathcal{K}_0 \star \left(\boldsymbol{H} - \overline{\boldsymbol{h}}_n^h\right), \nabla \times \left(\overline{\boldsymbol{h}}_n^h - \boldsymbol{H}\right)\right)$$
$$+ \int_0^\eta \left(\mathcal{K}_0 \star \left(\boldsymbol{H} - \overline{\boldsymbol{h}}_n^h\right), \nabla \times \left(\boldsymbol{h}_n^h - \overline{\boldsymbol{h}}_n^h\right)\right) + \int_0^\eta \left(\boldsymbol{f} - \overline{\boldsymbol{f}}_n, \widetilde{\boldsymbol{P}}_h\boldsymbol{H} - \boldsymbol{H}\right) + \int_0^\eta \left(\boldsymbol{f} - \overline{\boldsymbol{f}}_n, \boldsymbol{H} - \boldsymbol{h}_n^h\right) =: \sum_{i=1}^{8} S_i.$$

The following inequality is useful during the term by term estimation of the previous equality

$$\left\|\boldsymbol{h}_n^h(t) - \overline{\boldsymbol{h}}_n^h(t)\right\| \leqslant \tau\left\|\partial_t\boldsymbol{h}_n^h(t)\right\| \qquad \text{for } t \in [0, T].$$

Using Hölders inequality, Theorem 1 and Lemma 1(ii) give that

$$S_1 \leqslant \sqrt{\int_0^\eta \left\|\partial_t\boldsymbol{H} - \partial_t\boldsymbol{h}_n^h\right\|^2}\sqrt{\int_0^\eta \left\|\boldsymbol{H} - \widetilde{\boldsymbol{P}}_h\boldsymbol{H}\right\|^2} \lesssim \sqrt{\int_0^\eta \left\|\boldsymbol{H} - \widetilde{\boldsymbol{P}}_h\boldsymbol{H}\right\|^2}.$$

Again using Young's inequality gives

$$S_2 \leqslant \varepsilon \int_0^\eta \left\|\nabla \times \left(\boldsymbol{H} - \overline{\boldsymbol{h}}_n^h\right)\right\|^2 + C_\varepsilon \int_0^\eta \left\|\nabla \times \left(\boldsymbol{H} - \widetilde{\boldsymbol{P}}_h \boldsymbol{H}\right)\right\|^2.$$

Using Lemma 1(*ii*) gives

$$S_3 \leqslant \varepsilon \int_0^\eta \left\|\nabla \times \left(\boldsymbol{H} - \overline{\boldsymbol{h}}_n^h\right)\right\|^2 + C_\varepsilon \int_0^\eta \left\|\nabla \times \left(\boldsymbol{h}_n^h - \overline{\boldsymbol{h}}_n^h\right)\right\|^2 \leqslant \varepsilon \int_0^\eta \left\|\nabla \times \left(\boldsymbol{H} - \overline{\boldsymbol{h}}_n^h\right)\right\|^2 + C_\varepsilon \tau.$$

For the term S_4, we get

$$S_4 \overset{(4)}{\lesssim} \int_0^\eta \left\|\boldsymbol{H} - \overline{\boldsymbol{h}}_n^h\right\|^2 + \int_0^\eta \left\|\nabla \times \left(\boldsymbol{H} - \widetilde{\boldsymbol{P}}_h \boldsymbol{H}\right)\right\|^2.$$

Adding $\pm \boldsymbol{h}_n^h$ in the first term of the RHS of the inequality and employing Lemma 1(*ii*) gives

$$S_4 \lesssim \tau^2 + \int_0^\eta \left\|\boldsymbol{H} - \boldsymbol{h}_n^h\right\|^2 + \int_0^\eta \left\|\nabla \times \left(\boldsymbol{H} - \widetilde{\boldsymbol{P}}_h \boldsymbol{H}\right)\right\|^2.$$

In the same way as for the term S_4, we get thanks to Lemma 1(*ii*) that

$$S_5 \overset{(4)}{\leqslant} C_\varepsilon \int_0^\eta \left\|\boldsymbol{H} - \overline{\boldsymbol{h}}_n^h\right\|^2 + \varepsilon \int_0^\eta \left\|\nabla \times \left(\boldsymbol{H} - \overline{\boldsymbol{h}}_n^h\right)\right\|^2 \leqslant C_\varepsilon \tau^2 + C_\varepsilon \int_0^\eta \left\|\boldsymbol{H} - \boldsymbol{h}_n^h\right\|^2 + \varepsilon \int_0^\eta \left\|\nabla \times \left(\boldsymbol{H} - \overline{\boldsymbol{h}}_n^h\right)\right\|^2.$$

and

$$S_6 \overset{(4)}{\lesssim} \int_0^\eta \left\|\boldsymbol{H} - \overline{\boldsymbol{h}}_n^h\right\|^2 + \int_0^\eta \left\|\nabla \times \left(\boldsymbol{h}_n^h - \overline{\boldsymbol{h}}_n^h\right)\right\|^2 \lesssim \tau^2 + \int_0^\eta \left\|\boldsymbol{H} - \boldsymbol{h}_n^h\right\|^2 + \tau \lesssim \tau + \int_0^\eta \left\|\boldsymbol{H} - \boldsymbol{h}_n^h\right\|^2.$$

The terms S_7 and S_8 can be estimated due to the Lipschitz continuity of $\boldsymbol{f}$ by

$$S_7 \lesssim \int_0^\eta \left\|\boldsymbol{f} - \overline{\boldsymbol{f}}_n\right\|^2 + \int_0^\eta \left\|\widetilde{\boldsymbol{P}}_h \boldsymbol{H} - \boldsymbol{H}\right\|^2 \lesssim \tau^2 + \int_0^\eta \left\|\boldsymbol{H} - \widetilde{\boldsymbol{P}}_h \boldsymbol{H}\right\|^2$$

and

$$S_8 \lesssim \tau^2 + \int_0^\eta \left\|\boldsymbol{H} - \boldsymbol{h}_n^h\right\|^2.$$

Fixing a sufficiently small $\varepsilon > 0$, an application of the Grönwall argument concludes the proof.

(*ii*) The only difference with part (*i*) of the proof is the handling of the term S_1. Integration by parts gives

$$\begin{aligned} S_1 &= \left(\boldsymbol{H}(t) - \boldsymbol{h}_n^h(t), \boldsymbol{H}(t) - \widetilde{\boldsymbol{P}}_h \boldsymbol{H}(t)\right)\Big|_0^\eta - \int_0^\eta \left(\boldsymbol{H}(t) - \boldsymbol{h}_n^h(t), \partial_t \left(\boldsymbol{H}(t) - \widetilde{\boldsymbol{P}}_h \boldsymbol{H}(t)\right)\right) \\ &\leqslant \varepsilon \left\|\boldsymbol{H}(\eta) - \boldsymbol{h}_n^h(\eta)\right\|^2 + C_\varepsilon \left\|\boldsymbol{H}(\eta) - \widetilde{\boldsymbol{P}}_h \boldsymbol{H}(\eta)\right\|^2 + \left\|\boldsymbol{H}_0 - \widetilde{\boldsymbol{P}}_h \boldsymbol{H}_0\right\|^2 + C \int_0^\eta \left\|\boldsymbol{H} - \boldsymbol{h}_n^h\right\|^2 + C \int_0^\eta \left\|\partial_t \left(\boldsymbol{H} - \widetilde{\boldsymbol{P}}_h \boldsymbol{H}\right)\right\|^2. \end{aligned}$$

The rest of the proof follows closely the lines of (*i*).

(*iii*) The term τ^2 can be obtained by an application of Lemma 1(*iii*) instead of Lemma 1(*ii*) on the terms S_3 and S_6. □

4.1. Example: Nédélec's first family of curl-conforming finite elements of first order

Due to there practical importance, in the first example the lowest order Nédélec edge elements are considered [8]. The finite element space $\boldsymbol{V}^h$ is then given by

$$\boldsymbol{V}^h = \{\boldsymbol{v}^h \in \mathbf{H}(\mathbf{curl}, \Omega) \ : \ \boldsymbol{v}^h\big|_K(\mathbf{x}) = \boldsymbol{a}_K + \boldsymbol{b}_K \times \mathbf{x}, \quad \forall K \in \mathcal{T}^h\},$$

where $\boldsymbol{a}_K$ and $\boldsymbol{b}_K$ are constants in $\mathbb{R}^3$. The components of $\boldsymbol{a}_K$ and $\boldsymbol{b}_K$ are determined by the degrees of freedom $\int_e \boldsymbol{v}^h \cdot \hat{\boldsymbol{\tau}}$ on the six edges of a thetrahedron K with $\hat{\boldsymbol{\tau}}$ a unit vector along the edge e of K. Let us denote by $\boldsymbol{r}_h$ the interpolation operator valued in $\boldsymbol{V}_0^h$, defined element by element using $\boldsymbol{r}_h\boldsymbol{u}|_K = \boldsymbol{r}_K\boldsymbol{u}$ for all $K \in \mathcal{T}^h$, with $\boldsymbol{r}_K$ the element-wise interpolant given by

$$\int_e (\boldsymbol{u} - \boldsymbol{r}_K\boldsymbol{u}) \cdot \hat{\boldsymbol{\tau}} = 0, \qquad \text{for all edges } e \text{ of } K.$$

Unfortunately, the integrals appearing in this definition are not well defined for functions from $\mathbf{H}(\mathbf{curl}\,, \Omega)$. The interpolation operator $\boldsymbol{r}_h$ is defined in $\mathbf{H}^s(\mathbf{curl}\,, \Omega)$ for any $s > \frac{1}{2}$ [9, Lemma 5.1]. Moreover, there exists a constant $C > 0$, independent of h such that [9, Proposition 5.6]

$$\|\boldsymbol{H} - \boldsymbol{r}_h\boldsymbol{H}\| + \|\nabla \times (\boldsymbol{H} - \boldsymbol{r}_h\boldsymbol{H})\| \leqslant Ch^s \left(\|\boldsymbol{H}\|_{\mathbf{H}^s(\Omega)} + \|\nabla \times \boldsymbol{H}\|_{\mathbf{H}^s(\Omega)}\right),$$

for each $\boldsymbol{H} \in \mathbf{H}^s(\mathbf{curl}\,, \Omega)$ with $s \in \left(\frac{1}{2}, 1\right]$. Cea's lemma [10] implies that the projection operator $\widetilde{\boldsymbol{P}}_h$ defined in Section 4 for any $s \in \left(\frac{1}{2}, 1\right]$ has the property

$$\left\|\boldsymbol{u} - \widetilde{\boldsymbol{P}}_h\boldsymbol{u}\right\|_{\mathbf{H}(\mathbf{curl}\,,\Omega)} \leqslant \|\boldsymbol{u} - \boldsymbol{r}_h\boldsymbol{u}\|_{\mathbf{H}(\mathbf{curl}\,,\Omega)} \lesssim h^s\|\boldsymbol{u}\|_{\mathbf{H}^s(\mathbf{curl}\,,\Omega)}, \qquad \forall \boldsymbol{u} \in \mathbf{H}_0(\mathbf{curl}\,, \Omega) \cap \mathbf{H}^s(\mathbf{curl}\,, \Omega).$$

Now, the following corollary of Theorem 3 can be stated without proof.

Corollary 1. *Take* $s \in \left(\frac{1}{2}, 1\right]$. *Let* $\boldsymbol{f} \in \mathrm{Lip}([0, T], \mathbf{L}^2(\Omega))$.

(i) Suppose that $\boldsymbol{H}_0 \in \mathbf{H}_0(\mathbf{curl}\,, \Omega) \cap \mathbf{H}^s(\mathbf{curl}\,, \Omega)$ *and that the weak solution* $\boldsymbol{H}$ *of (2) satisfies*

$$\boldsymbol{H} \in \mathbf{L}^2\left((0, T), \mathbf{H}_0(\mathbf{curl}\,, \Omega) \cap \mathbf{H}^s(\mathbf{curl}\,, \Omega)\right).$$

Then there exists a constant C independent of both the time step τ *and the mesh size h such that*

$$\left\|\boldsymbol{H}(\eta) - \boldsymbol{h}_n^h(\eta)\right\|^2 + \int_0^\eta \left\|\nabla \times \left(\boldsymbol{H} - \overline{\boldsymbol{h}}_n^h\right)\right\|^2 \leqslant C\left(\tau + h^s\right)$$

is valid for any $\eta \in (0, T)$.

(ii) Suppose that $\boldsymbol{H}_0 \in \mathbf{H}_0(\mathbf{curl}\,, \Omega) \cap \mathbf{H}^s(\mathbf{curl}\,, \Omega)$ *and that the weak solution* $\boldsymbol{H}$ *of (2) satisfies*

$$\boldsymbol{H} \in \mathrm{H}^1\left((0, T), \mathbf{H}_0(\mathbf{curl}\,, \Omega) \cap \mathbf{H}^s(\mathbf{curl}\,, \Omega)\right).$$

Then there exists a constant C independent of both the time step τ *and the mesh size h such that*

$$\left\|\boldsymbol{H}(\eta) - \boldsymbol{h}_n^h(\eta)\right\|^2 + \int_0^\eta \left\|\nabla \times \left(\boldsymbol{H} - \overline{\boldsymbol{h}}_n^h\right)\right\|^2 \leqslant C\left(\tau + h^{2s}\right)$$

is valid for any $\eta \in (0, T)$.

(iii) If the initial condition satisfies $\nabla \times \boldsymbol{H}_0 \in \mathbf{H}_0(\mathbf{curl}\,, \Omega) \cap \mathbf{H}^s(\mathbf{curl}\,, \Omega)$ *and* $\mathcal{K}_0 \star \boldsymbol{H}_0 \in \mathbf{H}_0(\mathbf{curl}\,, \Omega) \cap \mathbf{H}^s(\mathbf{curl}\,, \Omega)$, *then the estimates in* (*i*) *and* (*ii*) *are satisfied with* τ^2 *instead of* τ.

Please note that the positive constant C in this estimates is of the form Ce^{CT}.

Thus, if $\tau \to 0$ and $h \to 0$, the convergence of the Rothe sequence $\boldsymbol{h}_n^h$ to the unique weak solution $\boldsymbol{H}$ of problem (2) in $C([0, T], \mathbf{L}^2(\Omega))$ is proved.

5. Higher regularity

The error estimates in the previous section have been obtained using a priori estimates, which were based on Grönwall's argument. The convergence rates are of order $O(\tau, h) = e^{CT}(\tau + h)$ in the space $C\left([0,T], \mathbf{L}^2(\Omega)\right)$ under appropriate conditions. To get rid of the exponential character of this constant, the use of Grönwall's lemma should be avoided. This can be done by incorporation of the curl operator $\nabla \times \boldsymbol{J}_s$ into a convolution kernel $\mathcal{K}$, see [1, Lemma 3], more specific

$$\nabla \times \boldsymbol{J}_s(\boldsymbol{x}, t) = -\int_\Omega \mathcal{K}(\boldsymbol{x}, \boldsymbol{x}')\boldsymbol{H}(\boldsymbol{x}', t)\,\mathrm{d}\boldsymbol{x}' =: -(\mathcal{K} \star \boldsymbol{H})(\boldsymbol{x}, t)$$

when $\boldsymbol{H}$ is divergence free and $\boldsymbol{H} \cdot \boldsymbol{\nu} = 0$ on $\partial\Omega$ (see also [3, §11.7] and [4]), where the kernel $\mathcal{K}$ is defined by

$$\mathcal{K} : \Omega \times \Omega \to \mathbb{R} : (\boldsymbol{x}, \boldsymbol{x}') \mapsto \kappa(|\boldsymbol{x} - \boldsymbol{x}'|), \text{ with } \kappa : (0, \infty) \to \mathbb{R} : s \mapsto \begin{cases} \frac{\widetilde{C}}{2s^2}\left(1 - \frac{s}{r_0}\right)\exp\left(-\frac{s}{r_0}\right) & s < r_0; \\ 0 & s \geqslant r_0. \end{cases}$$

Using the vector identity $-\Delta\boldsymbol{H} = \nabla \times (\nabla \times \boldsymbol{H}) - \nabla(\nabla \cdot \boldsymbol{H})$, the solution of problem (2) satisfies also

$$\begin{cases} \partial_t \boldsymbol{H} - \Delta\boldsymbol{H} + \mathcal{K} \star \boldsymbol{H} & = \boldsymbol{f} \quad \text{in } Q_T; \\ \boldsymbol{H} & = \boldsymbol{0} \quad \text{on } \partial\Omega \times (0,T); \\ \boldsymbol{H}(\boldsymbol{x}, 0) = \boldsymbol{H}_0, \quad \nabla \cdot \boldsymbol{H}_0 & = \boldsymbol{0} \quad \text{in } \Omega. \end{cases}$$

Therefore, under the additional assumption that $\boldsymbol{H} \cdot \boldsymbol{\nu} = 0$ on $\partial\Omega$, the solution to problem (2) obeys

$$(\partial_t \boldsymbol{H}, \boldsymbol{\varphi}) + (\nabla\boldsymbol{H}, \nabla\boldsymbol{\varphi}) + (\mathcal{K} \star \boldsymbol{H}, \boldsymbol{\varphi}) = (\boldsymbol{f}, \boldsymbol{\varphi}), \qquad \forall \boldsymbol{\varphi} \in \mathbf{H}^1_0(\Omega). \tag{12}$$

One major advantage of this formulation is the positive definiteness of the kernel $\mathcal{K}$ [1, Lemma 5]. It's this property, that makes it possible to avoid the use of Grönwall's lemma. Note also the following important estimates:

$$\mathcal{K}(\boldsymbol{x}, \cdot) \in L_p(\Omega) \text{ if } 1 \leqslant p < \frac{3}{2}, \forall \boldsymbol{x} \in \Omega.$$

and

$$|(\nabla \times \boldsymbol{J}_s)(\boldsymbol{x}, t)| = |(\mathcal{K} \star \boldsymbol{H})(\boldsymbol{x}, t)| = \left| \int_\Omega \mathcal{K}(\boldsymbol{x}, \boldsymbol{x}')\boldsymbol{H}(\boldsymbol{x}', t)\,\mathrm{d}\boldsymbol{x}' \right| \leqslant C(q)\,\|\boldsymbol{H}(t)\|_q, \qquad \forall q > 3, \qquad \forall \boldsymbol{x} \in \Omega. \tag{13}$$

Thanks to the Sobolev embeddings theorem in $\mathbb{R}^3$ holds that $\mathbf{H}^1_0(\Omega) \hookrightarrow \mathbf{L}^6(\Omega)$ [5, Thm. 3.6]. Employing this, together with the positive definiteness of $\mathcal{K}$ and Friedrichs inequality, gives for all $\boldsymbol{h}_1 \in \mathbf{H}^1_0(\Omega)$ and $\boldsymbol{h}_2 \in \mathbf{L}^2(\Omega)$ that

$$(\mathcal{K} \star \boldsymbol{h}_1, \boldsymbol{h}_2) \overset{(13)}{\leqslant} C_\varepsilon \|\boldsymbol{h}_1\|^2_{\mathbf{H}^1(\Omega)} + \varepsilon\|\boldsymbol{h}_2\|^2 \leqslant C_\varepsilon \|\nabla\boldsymbol{h}_1\|^2 + \varepsilon\|\boldsymbol{h}_2\|^2 \quad \text{and} \quad (\mathcal{K} \star \boldsymbol{h}_1, \boldsymbol{h}_1) \geqslant 0. \tag{14}$$

Again, a numerical scheme based on Backward Euler can be developed wherein the convolution is taken from the actual time step. The following results are obtained in [1]. The constants C are smaller in comparison with the constants appearing in Theorem 1 because Grönwall's argument is avoided.

Theorem 4 (Existence and uniqueness)**.**

- *Let $\boldsymbol{H}_0 \in \mathbf{L}^2(\Omega)$ and $\boldsymbol{f} \in L^2\left((0,T), \mathbf{L}^2(\Omega)\right)$. Assume that $\nabla \cdot \boldsymbol{H}_0 = 0 = \nabla \cdot \boldsymbol{f}(t)$ for any time $t \in [0,T]$. If $\boldsymbol{H} \cdot \boldsymbol{\nu} = 0$ on $\partial\Omega$, then the solution to problem (2) belongs to $C\left([0,T], \mathbf{L}^2(\Omega)\right) \cap L^2\left((0,T), \mathbf{H}^1_0(\Omega)\right)$ with $\partial_t \boldsymbol{H} \in L^2\left((0,T), \mathbf{H}^{-1}(\Omega)\right)$. If $\boldsymbol{H}_0 \in \mathbf{H}^1_0(\Omega)$, then $\partial_t\boldsymbol{H} \in L_2\left((0,T), \mathbf{L}^2(\Omega)\right)$.*
- *Moreover, assume that $\boldsymbol{f} \in \mathrm{Lip}\left([0,T], \mathbf{L}^2(\Omega)\right)$.*

  (i) If $\boldsymbol{H}_0 \in \mathbf{H}^1_0(\Omega)$ then

  $$\max_{t\in[0,T]} \|\boldsymbol{h}_n(t) - \boldsymbol{H}(t)\|^2 + \int_0^T \|\nabla[\boldsymbol{h}_n - \boldsymbol{H}]\|^2 \leqslant C\tau.$$

  (ii) If $\boldsymbol{H}_0 \in \mathbf{H}^1_0(\Omega) \cap \mathbf{H}^2(\Omega)$ then

  $$\max_{t\in[0,T]} \|\boldsymbol{h}_n(t) - \boldsymbol{H}(t)\|^2 + \int_0^T \|\nabla[\boldsymbol{h}_n - \boldsymbol{H}]\|^2 \leqslant C\tau^2.$$

5.1. A fully discrete finite element scheme

Now, $\boldsymbol{V}_0^h$ is a finite dimensional subspace of $\mathbf{H}_0^1(\Omega)$. The linear bounded operator $\overline{\boldsymbol{P}}_h : \mathbf{H}_0^1(\Omega) \to \boldsymbol{V}_0^h$ is defined such that if $\boldsymbol{u} \in \mathbf{H}_0^1(\Omega)$, then $\overline{\boldsymbol{P}}_h\boldsymbol{u} \in \boldsymbol{V}_0^h$ satisfies $(\boldsymbol{u}, \boldsymbol{v}_h) + (\nabla\boldsymbol{u}, \nabla\boldsymbol{v}_h) = \left(\overline{\boldsymbol{P}}_h\boldsymbol{u}, \boldsymbol{v}_h\right) + \left(\nabla\overline{\boldsymbol{P}}_h\boldsymbol{u}, \nabla\boldsymbol{v}_h\right)$ for all $\boldsymbol{v}_h \in \boldsymbol{V}_0^h$. The following fully discrete linear recurrent scheme is proposed: find $\boldsymbol{h}_i^h \in \boldsymbol{V}_0^h$ such that

$$\begin{cases} \left(\delta\boldsymbol{h}_i^h, \boldsymbol{\varphi}^h\right) + \left(\nabla\boldsymbol{h}_i^h, \nabla\boldsymbol{\varphi}^h\right) + \left(\mathcal{K} \star \boldsymbol{h}_i^h, \boldsymbol{\varphi}^h\right) & = \left(\boldsymbol{P}_h\boldsymbol{f}_i, \boldsymbol{\varphi}^h\right) = \left(\boldsymbol{f}_i, \boldsymbol{\varphi}^h\right); \\ \boldsymbol{h}_0^h & = \overline{\boldsymbol{P}}_h\boldsymbol{H}_0, \end{cases} \tag{15}$$

is satisfied for all $\boldsymbol{\varphi}^h \in \boldsymbol{V}_0^h$. Due to the positive definiteness of $\mathcal{K}$, an application of the Lax-Milgram lemma gives the existence of a unique solution in $\boldsymbol{V}_0^h$ of (15) for any $i = 1, \ldots, n$ and any $\tau > 0$ if $\boldsymbol{H}_0 \in \mathbf{H}_0^1(\Omega)$.

The same stability results are obtained as in Lemma 1, where the curl-spaces are replaced by analogous $\mathbf{H}^s(\Omega)$-spaces. Now, the use of Grönwall's argument is avoided.

Lemma 2 (Enhanced stability)**.** *Assume that $\boldsymbol{f} \in L^2\left((0,T), \mathbf{L}^2(\Omega)\right)$, $\nabla \cdot \boldsymbol{H}_0 = 0 = \nabla \cdot \boldsymbol{f}(t)$ for any time $t \in [0,T]$ and $\boldsymbol{H} \cdot \boldsymbol{\nu} = 0$ on $\partial\Omega$.*

(i) Let $\boldsymbol{H}_0 \in \mathbf{H}_0^1(\Omega)$. Then, there exists a positive constant C such that for all $\tau > 0$

$$\max_{1\leqslant i\leqslant n} \left\|\boldsymbol{h}_i^h\right\|^2 + \sum_{i=1}^n \left\|\boldsymbol{h}_i^h - \boldsymbol{h}_{i-1}^h\right\|^2 + \sum_{i=1}^n \left\|\nabla\boldsymbol{h}_i^h\right\|^2 \tau \leqslant C.$$

(ii) If $\boldsymbol{H}_0 \in \mathbf{H}_0^1(\Omega)$, then for all $\tau > 0$

$$\max_{1\leqslant i\leqslant n} \left\|\nabla\boldsymbol{h}_i^h\right\|^2 + \sum_{i=1}^n \left\|\nabla\boldsymbol{h}_i^h - \nabla\boldsymbol{h}_{i-1}^h\right\|^2 + \sum_{i=1}^n \left\|\delta\boldsymbol{h}_i^h\right\|^2 \tau \leqslant C.$$

(iii) If $\boldsymbol{f}(0) \in \mathbf{L}^2(\Omega)$, $\partial_t\boldsymbol{f} \in L^2\left((0,T), \mathbf{L}^2(\Omega)\right)$ and $\boldsymbol{H}_0 \in \mathbf{H}_0^1(\Omega) \cap \mathbf{H}^2(\Omega)$ then for all $\tau > 0$

$$\max_{1\leqslant i\leqslant n} \left\|\delta\boldsymbol{h}_i^h\right\|^2 + \sum_{i=1}^n \left\|\delta\boldsymbol{h}_i^h - \delta\boldsymbol{h}_{i-1}^h\right\|^2 + \sum_{i=1}^n \left\|\nabla\delta\boldsymbol{h}_i^h\right\|^2 \tau \leqslant C.$$

Proof. (i) Set $\boldsymbol{\varphi}^h = \boldsymbol{h}_i^h$ in (15). Multiply the result by τ and sum up for $i = 1, \ldots, j$ to get

$$\sum_{i=1}^j \left(\delta\boldsymbol{h}_i^h, \boldsymbol{h}_i^h\right)\tau + \sum_{i=1}^j \left\|\nabla\boldsymbol{h}_i^h\right\|^2 \tau + \sum_{i=1}^j \left(\mathcal{K} \star \boldsymbol{h}_i^h, \boldsymbol{h}_i^h\right)\tau = \sum_{i=1}^j \left(\boldsymbol{f}_i, \boldsymbol{h}_i^h\right)\tau.$$

The use of Grönwall's argument can be avoided by employing the positive definiteness of $\mathcal{K}$ and Friedrichs inequality. Indeed, it holds that $\sum_{i=1}^j \left(\mathcal{K} \star \boldsymbol{h}_i^h, \boldsymbol{h}_i^h\right)\tau \geqslant 0$ and

$$\left|\sum_{i=1}^j \left(\boldsymbol{f}_i, \boldsymbol{h}_i^h\right)\tau\right| \leqslant C_\varepsilon \sum_{i=1}^j \left\|\boldsymbol{f}_i\right\|^2 \tau + \varepsilon \sum_{i=1}^j \left\|\boldsymbol{h}_i^h\right\|^2 \tau \leqslant C_\varepsilon \sum_{i=1}^j \left\|\boldsymbol{f}_i\right\|^2 \tau + \varepsilon \sum_{i=1}^j \left\|\nabla\boldsymbol{h}_i^h\right\|^2 \tau.$$

Fixing ε sufficiently small gives the proof.

(ii) We put $\boldsymbol{\varphi}^h = \delta\boldsymbol{h}_i^h$ in (15). Again, we multiply by τ and sum up for $i = 1, \ldots, j$

$$\sum_{i=1}^j \left\|\delta\boldsymbol{h}_i^h\right\|^2 \tau + \sum_{i=1}^j \left(\nabla\boldsymbol{h}_i^h, \nabla\boldsymbol{h}_i^h - \nabla\boldsymbol{h}_{i-1}^h\right) + \sum_{i=1}^j \left(\mathcal{K} \star \boldsymbol{h}_i^h, \delta\boldsymbol{h}_i^h\right)\tau = \sum_{i=1}^j \left(\boldsymbol{f}_i, \delta\boldsymbol{h}_i^h\right)\tau.$$

Using (i), we obtain

$$\left|\sum_{i=1}^j \left(\mathcal{K} \star \boldsymbol{h}_i^h, \delta\boldsymbol{h}_i^h\right)\tau\right| \overset{(14)}{\leqslant} C_\varepsilon \sum_{i=1}^j \left\|\nabla\boldsymbol{h}_i^h\right\|^2 \tau + \varepsilon \sum_{i=1}^j \left\|\delta\boldsymbol{h}_i^h\right\|^2 \tau \leqslant C_\varepsilon + \varepsilon \sum_{i=1}^j \left\|\delta\boldsymbol{h}_i^h\right\|^2 \tau.$$

(iii) The proof is the same as in Lemma 1(iii). Now, the following compatibility condition is needed

$$\delta \boldsymbol{h}_0^h := \boldsymbol{P}_h \boldsymbol{f}(0) - \boldsymbol{P}_h \left(\Delta \boldsymbol{H}_0\right) - \boldsymbol{P}_h \left(\mathcal{K} \star \boldsymbol{H}_0\right).$$

$\square$

Using the Rothe's functions, the variational formulation (15) can be rewritten as

$$\left(\partial_t \boldsymbol{h}_n^h(t), \boldsymbol{\varphi}^h\right) + \left(\nabla \overline{\boldsymbol{h}}_n^h(t), \nabla \boldsymbol{\varphi}^h\right) + \left(\mathcal{K} \star \overline{\boldsymbol{h}}_n^h(t), \boldsymbol{\varphi}^h\right) = \left(\overline{\boldsymbol{f}}_n(t), \boldsymbol{\varphi}^h\right), \qquad \boldsymbol{\varphi}^h \in \boldsymbol{V}_0^h. \tag{16}$$

The following error estimates have smaller constant C in comparison with the constants appearing in Theorem 3 because Grönwall's argument is avoided thanks to the positive definiteness of $\mathcal{K}$.

Theorem 5. *Suppose that* $\boldsymbol{f} \in \mathrm{Lip}([0,T], \mathbf{L}^2(\Omega))$, $\nabla \cdot \boldsymbol{H}_0 = 0 = \nabla \cdot \boldsymbol{f}(t)$ *for any time* $t \in [0,T]$ *and* $\boldsymbol{H} \cdot \boldsymbol{\nu} = 0$ *on* $\partial\Omega$.

(i) Let the weak solution $\boldsymbol{H}$ *of (2) at time t and the initial condition* $\boldsymbol{H}_0$ *satisfy* $\boldsymbol{H}(t), \boldsymbol{H}_0 \in \mathbf{H}_0^1(\Omega)$. *Then for any* $\tau < \tau_0$, *there exists a constant C independent of both the time step* τ *and the mesh size h, such that*

$$\begin{aligned}
&\left\|\boldsymbol{H}(\eta) - \boldsymbol{h}_n^h(\eta)\right\|^2 + \int_0^\eta \left\|\nabla\left(\boldsymbol{H} - \overline{\boldsymbol{h}}_n^h\right)\right\|^2 \\
&\quad \leqslant C\left(\tau + \left\|\boldsymbol{H}_0 - \overline{\boldsymbol{P}}_h \boldsymbol{H}_0\right\|^2 + \sqrt{\int_0^\eta \left\|\boldsymbol{H} - \overline{\boldsymbol{P}}_h \boldsymbol{H}\right\|^2} + \int_0^\eta \left\|\boldsymbol{H} - \overline{\boldsymbol{P}}_h \boldsymbol{H}\right\|^2 + \int_0^\eta \left\|\nabla\left(\boldsymbol{H} - \overline{\boldsymbol{P}}_h \boldsymbol{H}\right)\right\|^2\right),
\end{aligned}$$

is valid for any $\eta \in (0,T)$*;*

(ii) Let the weak solution $\boldsymbol{H}$ *of (2) at time t and the initial condition* $\boldsymbol{H}_0$ *satisfy* $\boldsymbol{H}(t), \partial_t \boldsymbol{H}(t), \boldsymbol{H}_0 \in \mathbf{H}_0^1(\Omega)$. *Then for any* $\tau < \tau_0$, *there exists a constant C independent of both the time step* τ *and the mesh size h, such that*

$$\begin{aligned}
\left\|\boldsymbol{H}(\eta) - \boldsymbol{h}_n^h(\eta)\right\|^2 + \int_0^\eta \left\|\nabla\left(\boldsymbol{H} - \overline{\boldsymbol{h}}_n^h\right)\right\|^2 &\leqslant C\Big(\tau + \left\|\boldsymbol{H}_0 - \overline{\boldsymbol{P}}_h \boldsymbol{H}_0\right\|^2 + \left\|\boldsymbol{H}(\eta) - \overline{\boldsymbol{P}}_h \boldsymbol{H}(\eta)\right\|^2 \\
&\quad + \int_0^\eta \left\|\boldsymbol{H} - \overline{\boldsymbol{P}}_h \boldsymbol{H}\right\|^2 + \int_0^\eta \left\|\partial_t\left(\boldsymbol{H} - \overline{\boldsymbol{P}}_h \boldsymbol{H}\right)\right\|^2 + \int_0^\eta \left\|\nabla\left(\boldsymbol{H} - \overline{\boldsymbol{P}}_h \boldsymbol{H}\right)\right\|^2\Big),
\end{aligned}$$

is valid for any $\eta \in (0,T)$*;*

(iii) If the initial condition satisfies $\boldsymbol{H}_0 \in \mathbf{H}_0^1(\Omega) \cap \mathbf{H}^2(\Omega)$, *then the estimates in* (i) *and* (ii) *are satisfied with* τ^2 *instead of* τ.

Proof. (i) We subtract (16) from (12) for $\boldsymbol{\varphi} = \boldsymbol{\varphi}^h$. We set $\boldsymbol{\varphi}^h = \overline{\boldsymbol{P}}_h \boldsymbol{H}(t) - \boldsymbol{h}_n^h(t)$ and integrate in time over $(0,\eta)$ for $\eta \in [0,T]$ and rearrange the terms to obtain

$$\begin{aligned}
&\frac{1}{2}\left\|\boldsymbol{H}(\eta) - \boldsymbol{h}_n^h(\eta)\right\|^2 - \frac{1}{2}\left\|\boldsymbol{H}_0 - \overline{\boldsymbol{P}}_h \boldsymbol{H}_0\right\|^2 + \int_0^\eta \left\|\nabla\left(\boldsymbol{H} - \overline{\boldsymbol{h}}_n^h\right)\right\|^2 + \int_0^\eta \left(\mathcal{K} \star \left(\boldsymbol{H} - \overline{\boldsymbol{h}}_n^h\right), \boldsymbol{H} - \overline{\boldsymbol{h}}_n^h\right) \\
&\quad = \int_0^\eta \left(\partial_t \boldsymbol{H} - \partial_t \boldsymbol{h}_n^h, \boldsymbol{H} - \overline{\boldsymbol{P}}_h \boldsymbol{H}\right) + \int_0^\eta \left(\nabla\left(\boldsymbol{H} - \overline{\boldsymbol{h}}_n^h\right), \nabla\left(\boldsymbol{H} - \overline{\boldsymbol{P}}_h \boldsymbol{H}\right)\right) + \int_0^\eta \left(\nabla\left(\boldsymbol{H} - \overline{\boldsymbol{h}}_n^h\right), \nabla\left(\boldsymbol{h}_n^h - \overline{\boldsymbol{h}}_n^h\right)\right) \\
&\qquad + \int_0^\eta \left(\mathcal{K} \star \left(\boldsymbol{H} - \overline{\boldsymbol{h}}_n^h\right), \boldsymbol{H} - \overline{\boldsymbol{P}}_h \boldsymbol{H}\right) + \int_0^\eta \left(\mathcal{K} \star \left(\boldsymbol{H} - \overline{\boldsymbol{h}}_n^h\right), \boldsymbol{h}_n^h - \overline{\boldsymbol{h}}_n^h\right) \\
&\qquad + \int_0^\eta \left(\boldsymbol{f} - \overline{\boldsymbol{f}}_n, \overline{\boldsymbol{P}}_h \boldsymbol{H} - \boldsymbol{H}\right) + \int_0^\eta \left(\boldsymbol{f} - \overline{\boldsymbol{f}}_n, \boldsymbol{H} - \boldsymbol{h}_n^h\right) =: \sum_{i=1}^{7} S_i.
\end{aligned}$$

The terms S_1, S_2, S_3, S_6 and S_7 can be handled in the same way as in Theorem 3. For the others terms, we get that

$$S_4 \overset{(14)}{\leqslant} \varepsilon \int_0^\eta \left\|\nabla\left(\boldsymbol{H} - \overline{\boldsymbol{h}}_n^h\right)\right\|^2 + C_\varepsilon \int_0^\eta \left\|\boldsymbol{H} - \overline{\boldsymbol{P}}_h \boldsymbol{H}\right\|^2.$$

and

$$S_5 \overset{(14)}{\leqslant} \varepsilon \int_0^\eta \left\| \nabla \left(\boldsymbol{H} - \overline{\boldsymbol{h}}_n^h \right) \right\|^2 + C_\varepsilon \int_0^\eta \left\| \boldsymbol{h}_n^h - \overline{\boldsymbol{h}}_n^h \right\|^2 \leqslant \varepsilon \int_0^\eta \left\| \nabla \left(\boldsymbol{H} - \overline{\boldsymbol{h}}_n^h \right) \right\|^2 + C_\varepsilon \tau.$$

Fixing a sufficiently small $\varepsilon > 0$ concludes the proof.

(*ii*) and (*iii*) The proof follows the same lines as in Theorem 3(*ii*) and (*iii*). □

5.2. Example: Lagrangian finite elements

In this example, the first-order Lagrange finite elements for the space discretization are considered. The finite element space $\boldsymbol{V}^h$ is now given by $\boldsymbol{V}^h = \{\boldsymbol{v}^h \in \mathbf{H}^1(\Omega) : \boldsymbol{v}^h\big|_K \in \boldsymbol{P}_1(K), \quad \forall K \in \mathcal{T}^h\}$, with $\boldsymbol{P}_1(K)$ the space of componentwise first-order polynomials. The coefficients of this polynomials are determined by the degrees of freedom $\boldsymbol{v}^h(\mathbf{a}_i)$ with $\mathbf{a}_i, i = 1, \ldots, 4$, the vertices of K. Note that $\boldsymbol{V}_0^h = \{\boldsymbol{v}^h \in \boldsymbol{V}^h : \boldsymbol{v}^h = \mathbf{0} \text{ on } \partial\Omega\}$. The corresponding interpolation operator is denoted by $\boldsymbol{\pi}_h$. The Sobolev Embedding theorem in $\mathbb{R}^3$ [11, Theorem 7.57] implies that $\mathbf{H}^s(\Omega) \subset \boldsymbol{C}(\overline{\Omega})$ if $s > \frac{3}{2}$. Thus $\boldsymbol{\pi}_h : \mathbf{H}^s(\Omega) \to \boldsymbol{V}_0^h, s > \frac{3}{2}$, to ensure that the vertex values are well defined. Then, [5, Theorem 5.48] gives that there exists a constant $C > 0$ independent of h such that

$$\|\boldsymbol{u} - \boldsymbol{\pi}_h \boldsymbol{u}\|_{\mathbf{H}^1(\Omega)} \leqslant C h^{s-1} \|\boldsymbol{u}\|_{\mathbf{H}^s(\Omega)}, \tag{17}$$

for each $\boldsymbol{u} \in \mathbf{H}^s(\Omega)$ with $\frac{3}{2} < s \leqslant 2$. The paper finishes with the following corollary.

Corollary 2. *Take $s \in \left(\frac{3}{2}, 2\right]$. Let $\boldsymbol{f} \in \operatorname{Lip}\left([0,T], \mathbf{L}^2(\Omega)\right)$, $\nabla \cdot \boldsymbol{H}_0 = 0 = \nabla \cdot \boldsymbol{f}(t)$ for any time $t \in [0,T]$ and $\boldsymbol{H} \cdot \boldsymbol{\nu} = 0$ on $\partial\Omega$.*

(i) Suppose that $\boldsymbol{H}_0 \in \mathbf{H}_0^1(\Omega) \cap \mathbf{H}^s(\Omega)$ and that the weak solution $\boldsymbol{H}$ of (2) satisfies

$$\boldsymbol{H} \in \mathbf{L}^2\left((0,T), \mathbf{H}_0^1(\Omega) \cap \mathbf{H}^s(\Omega)\right).$$

Then the error estimate

$$\left\|\boldsymbol{H}(\eta) - \boldsymbol{h}_n^h(\eta)\right\|^2 + \int_0^\eta \left\| \nabla \left(\boldsymbol{H} - \overline{\boldsymbol{h}}_n^h \right) \right\|^2 \lesssim \tau + h^{s-1},$$

is valid for any $\eta \in (0,T)$.

(ii) Suppose that $\boldsymbol{H}_0 \in \mathbf{H}_0^1(\Omega) \cap \mathbf{H}^s(\Omega)$ and that the weak solution $\boldsymbol{H}$ of (2) satisfies

$$\boldsymbol{H} \in \mathrm{H}^1\left((0,T), \mathbf{H}_0^1(\Omega) \cap \mathbf{H}^s(\Omega)\right).$$

Then the error estimate

$$\left\|\boldsymbol{H}(\eta) - \boldsymbol{h}_n^h(\eta)\right\|^2 + \int_0^\eta \left\| \nabla \left(\boldsymbol{H} - \overline{\boldsymbol{h}}_n^h \right) \right\|^2 \lesssim \tau + h^{2(s-1)},$$

is valid for any $\eta \in (0,T)$.

(iii) If the initial condition satisfies $\nabla \times \boldsymbol{H}_0 \in \mathbf{H}_0^1(\Omega) \cap \mathbf{H}^s(\Omega)$ and $\mathcal{K}_0 \star \boldsymbol{H}_0 \in \mathbf{H}_0^1(\Omega) \cap \mathbf{H}^s(\Omega)$, then the estimates in (i) and (ii) are satisfied with τ^2 instead of τ.

6. Conclusion

In this contribution, the convergence of a fully discrete finite element scheme (10) to the solution of problem (2) is shown. Moreover, it is demonstrated how to improve the error estimates under higher regularity.

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